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Involving nine school systems, the intent of the project was to provide students with experiences that reflected a valid interpretation of good mathematical thinking, to make extensive data collections to document the nature of the experiences, to achieve the most effective possible quality control in assuring the relevance and effectiveness of the experiences, and to integrate this demonstration into the project's program of in-service teacher training and the development of curriculum materials. An organizational structure between the Project and the participating school systems was developed to provide for competent staffing of each school and to arrange for Project materials to be taught to students in grades 4 through 7 for one hour weekly. Teachers were provided with weekly or bi-weekly in-service training. Systematic, formal instruction was not included; instead, a problem was presented and the class, working as a group, solved it the best way they could. Some children emerged as leaders, while others chose not to participate and were not forced to do so. Children had a measure of freedom to think about a problem and to solve it. After an analysis of first year data (of the 5 year project), indications are that creative use of mathematical ideas is strongly dependent upon school experiences; that under certain conditions a significant fraction of children from grades 4 through 7 can profit from the ideas presented; that integration of exploratory mathematics into the school program can be achieved with reasonable success; and that in-service teacher education is very helpful.

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A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE

COOPERATIVE RESEARCH PROJECT #D-022

ROBERT B. DAVIS

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A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE. (THE MADISON PROJECT)

Cooperative Research Project No. D-022

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Syracuse University - Webster College
Syracuse 10, New York - St. Louis 19, Missouri

1962

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ABSTRACT

A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE. (THE MADISON PROJECT)

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Cooperative Research Project No. D-022

1 September 1961 - 31 October 1962

BACKGROUND

The recent past has witnessed widespread activity in relation to "new" mathematics curricula for grades K-12. The central essence of "modernization" efforts probably lies in a "new" interpretation of the nature of mathematical thought.[1] "Mathematical thought" as interpreted by mathematicians in a perishable intangible not easily taught on a broad scale; it differs markedly from the more tangible set of techniques and facts that usually received emphasis in "traditional" curricula. Considerable informal evidence points to the difficulties encountered in attempting widespread teaching of "modern" mathematical thinking, and to the consequent need for careful regulation of the mathematical experiences offered by the school. The present "demonstration" is an attempt to work with approximately 9 school systems, in providing for students some experiences that will reflect a valid interpretation of "good mathematical thinking."

Footnote 1: The use of the words "new" and "modern" in relation to recent attempts to revise the mathematics curriculum is inaccurate on several counts; nonetheless the usage has become almost universal, and these words are used here for want of satisfactory alternatives. A similar situation exists in relation to the word "traditional."

OBJECTIVES

The objectives of the "demonstration" are:

1) To work with approximately 9 school systems in providing students with experiences that accurately reflect a valid interpretation of "good mathematical thinking";

2) To make extensive collection of data that will serve to document, as specifically as possible, the nature of these experiences;

3) To achieve the most effective possible "quality control" in assuring the relevance and effectiveness of these experiences;

4) To integrate this "demonstration" as effectively as possible into the Madison Project's program of in-service teacher training, pre-service teacher education, and development of curriculum materials;

5) To report as effectively as possible to the profession and to the academic community: that is, to teachers, principals, supervisors, other educators, mathematicians, psychologists, scientists, board of education members, parents, etc. (It is assumed that this communication will be in the nature of a dialogue, in which the Project will receive valuable counsel from the academic community, particularly concerning the effectiveness of Project materials and the general appropriateness of the learning experiences.)

PROCEDURE

It should be recognized that the present "demonstration" is an exercise in evolution -- it is not conceived to test explicit hypotheses, but rather to develop a combination of various attributes (not fully explicit) that together will serve to give children experience in good mathematical thinking. Furthermore, for the "demonstration" project, the academic year 1961-1962 has been the first year of a proposed five year development.

The procedure during 1961-1962 has consisted of the following:

1) Developing an organizational structure between the Project and the participating school systems that provides for mathematically and pedagogically competent personnel to supervise work in each participating school;

2) Arrange for Project materials to be taught to children in grades 4, 5, 6, and 7, for approximately one hour per week;

3) (The development of Project materials has been supported by other sponsoring agencies, namely: the Marcel Holzer Foundation, the Alfred P. Sloan Foundation, the Ford Foundation, the National Science Foundation, and Syracuse University);

4) Provide weekly or bi-weekly in-service training for teachers in participating schools (some of this activity has been sponsored by the National Science Foundation);

5) Provide more intensive training for the Project's "demonstration teachers";

6) Accumulate a library of tape-recordings of actual classroom lessons;

7) Arrange for a panel of mathematicians, teachers, and psychologists to analyze selected tape-recorded lesson excerpts for purposes of an analytical identification of apparently relevant factors in the mathematical thinking, the classroom experience, and teacher effectiveness;

8) Begin the use of the panel analysis just mentioned as a means of determining the effectiveness of the classroom experiences provided for the children;

9) To accumulate a library of 16 mm. sound motion pictures recording actual classroom lessons, as a means of preserving the fullest possible record of the nature of the classroom experiences that are provided for the children (the cost of this was met mainly by a group of industries and trade unions in greater St. Louis);

10) Use of the films just mentioned as a means of reporting to the teaching and mathematics professions;

11) Use of the films just mentioned as a means of teacher training;

12) Extensive consultation with professional leaders as a guarantee of the relevance and appropriateness of the learning experiences provided by the Project.

RESULTS AND CONCLUSIONS

Recalling that 1961-1962 was the first year of a proposed 5 year activity, it would be premature to attempt a report of "conclusive" results at this time.

Nonetheless, analysis of tape recordings, films and school-administered achievement tests indicates that:

1) Under certain conditions a significant fraction of 4th, 5th, 6th, and 7th graders can make creative use of the mathematical ideas presented in the Madison Project materials;

2) Creative use of these mathematical ideas by the children is strongly dependent upon the nature of the experiences provided by the school;

3) At least some of the relevant aspects of these experiences can be identified by the study of tape-recorded lesson excerpts, and

can be communicated in highly specific form through the medium of such tape recordings;

4) In-service teacher education can help teachers to provide children with more satisfactory experiences in creative mathematical exploration;

5) Integration of such "creative exploratory mathematical experiences" into the school program can be achieved with reasonable success, as judged by teachers and administrators;

6) Integration of such experiences into the school program shows no observable damage to other parts of the child's educational experiences, as judged from school-administered achievement tests, observations of school psychologists, and the judgment of teachers and principals;

7) The use of films as a means of recording classroom experiences appears to have many important advantages over most other methods of data recording.

It should be noted that the response of students, parents, and communities generally has been most enthusiastic. Parents regularly report that the weekly "mathematics lesson" has become a high-point of many children's school experiences. In addition,

eminent mathematicians who have studied Project films have strongly endorsed the general appropriateness and value of the kinds of mathematical experiences that are being provided for the children. Much of this response in both cases probably does not merit serious quantification, but nonetheless does serve as an indicator where decisions are primarily a matter of value judgments.

Many problems of teacher training and many measurement problems remain unsolved, and the widespread establishment of effective teaching of Project materials has not yet been fully realized. Such progress in these directions as may be possible must remain the task of the succeeding four years of Project activity. The second year of Project activity in regard to the present "demonstration" is now vigorously underway.

I. THE STATE OF THE MADISON PROJECT AT THE OUTSET

The "demonstration" described here, and supported as U. S. Office of Education Demonstration Project No. D-022, operated mainly during the academic year September 1961 - August 1962. Since it represents a portion of the continuing operation of the Syracuse University - Webster College "Madison Project," its purposes and methods can best be understood by looking first at the state of the Madison Project immediately prior to the beginning of this period -- that is, during the early months of 1961.

The Place of the Project in the School Program. The Project had for several years been engaged in developing some informal exploratory experiences in algebra and geometry, suitable for presentation to children in (approximately) grades[1] 2 through 8. The study of Project material was regarded as supplementary to elementary school programs in arithmetic and in science, and not as a replacement for such programs. The experimental development of the Project materials took place in several schools, but mainly in the public elementary school in the town of Weston, Connecticut. Project personnel included Associate Professor Robert B. Davis, of Syracuse University, who served as Director of the Project and who was in charge of developing Project materials; Mrs. Beryl Cochran, co-ordinator for the Weston experimental center; Professor Donald E. Kibbey, co-ordinator for Syracuse University; Morton Schindel, technical advisor for audio-visual and tape-recording activities; Mrs. Marilyn Hurley, administrative assistant; a number of mathematicians, psychologists, physicists, engineers, and educators who served as advisors and consultants; and a staff of clerical and secretarial workers.

Informal Exploratory Experiences

Since the concept of "informal exploratory experiences" for children plays an essential role in what follows, it is advisable to clarify this notion somewhat immediately.

1) "Traditional" school programs (as of, say, 1950 - 1960) quite commonly provided excellent "informal exploratory experiences" with whole numbers (i.e., 1, 2, 3, 4, ...) to children in kindergarten and grades 1, 2, and 3. These "informal exploratory experiences" took such

Footnote 1: Actually, the Project materials are not tied to specific grade levels, and several of the schools in which the Project operates have "ungraded" elementary schools. References to "grade level" are consequently very approximate and flexible.

forms as:

keeping track of the day of the month,
taking attendance,
determining the number of children absent,
determining the number of boys absent,
determining the number of girls absent,
counting milk money,
counting pencils, and determining whether there were enough for each child to receive one; and how many would be left over after each child did receive one,
keeping score in various simple games,
finding "how many days until Christmas,"
counting by two's,
measuring things with a ruler (perhaps to the nearest inch),

and so on.

An important aspect of these "informal exploratory experiences" is that systematic formal instruction was not included. A problem was presented to the class -- such as the problem of counting up the milk money -- and the class, working together as a group, solved it as best they could. Quite commonly the class activity had these features:

i) The teacher did not tell the students how to solve the problem;
ii) Some students emerged as "leaders" who were able to solve the problem;
iii) At least initially, some students did not participate in solving the problem; these children were not forced into participation. There was, however, a hope that the class activity would prove attractive enough to entice these children into becoming interested in it;
iv) At least initially, there was no required minimum standard of achievement;
v) Children "solved" these problems by a variety of methods; the child had at least some measure of freedom to think about a problem by methods of his own devising.

2) A more definitive description of what is meant by "informal exploratory experiences" than that given above is clearly desirable. Such a description, to have any real value, must be sufficiently detailed to be useful in making decisions about actual classroom teaching, yet at the same time it must allow some variation for individual differences

in the "teaching style" of different teachers. It must deal with essential aspects of teaching, and not superficial ones. (This last is a stringent requirement, since lessons "superficially similar" may in fact be basically quite different: for example, one teacher says "Let's try to do ___," and her words are an invitation; another teacher says "Let's try to do ___," and her words are a strict and harsh command. The superficial similarity of word choice must not be allowed to obscure the fact that the child is confronted with entirely dissimilar situations in these two cases.)

We shall return later in this report to the problem of describing what is properly included in the category of "informal exploratory experiences."

3) We have considered "informal exploratory experiences" in relation to whole numbers (including situations related to adding, subtracting, etc.), as provided in grades K-3 or thereabouts. Where else in the school program are such experiences provided? Where are they conspicuous by their absence? Where are they actually desirable? What advantages can be expected to result from their inclusion? How important a role can they play in the mathematics curriculum?

In the next few sections we shall take a preliminary look at some of these questions:

4) The case of fractions usually provides -- in the "traditional" program -- a contrasting situation. Formal instruction in dealing with fractions is included in, perhaps, grades 4 - 6 (or so). At this time children are told how to add, subtract, multiply, and divide fractions, and are given formal instruction in various uses and various meanings of fractions.

This work is not (traditionally) accompanied or preceded by extensive "informal exploratory experiences." It has long been clear that children do not feel the same confidence in dealing with fractions that they feel when dealing with whole numbers. This evidently raises the question of whether it might be possible to devise an extensive program of "informal exploratory experiences" with fractions that could result in children dealing with fractions more confidently and more creatively than they usually do at present. The beginnings of success in devising such experiences would, in turn, raise several further questions -- particularly in relation to grade placement of these informal experiences, and of whatever formal instruction might follow them.

5) The case of the "traditional" ninth-grade algebra course is particularly relevant to Madison Project activity. The "traditional" ninth-grade algebra course has come under considerable criticism for several reasons, including the fact that it was intrinsically pre-occupied with the manipulation of relatively meaningless symbols.[2] Another troublesome aspect of ninth-grade algebra in the "traditional" curriculum was that it was not preceded[3] by very much prior "informal exploratory experience."

As a result, the student in ninth-grade algebra was confronted suddenly -- and simultaneously -- with new ideas, unfamiliar notations, new definitions, a new general approach to mathematics, new kinds of problems, and new techniques. Too much of this had too little meaning, and algebra became for many students a matter of unrewarding complexity and unexciting mystery.

This, in turn, led to a need to re-teach much of ninth-grade algebra again in grade eleven, and often still again in college freshman mathematics.

6) Suppose, instead, a large amount of "informal exploratory experience" in algebra (and related aspects of geometry) could be provided in grades 2-8. Would this have any marked effect on the learning of algebra in grades 8 and 9? How far into algebra could children penetrate through such "informal exploratory experience"? Would it be possible to replace traditional secondary mathematics by quite different courses -- and mathematically stronger courses -- once students entered grade 8 with this new kind of background? How should the transition from "informal exploratory experience" to formal instruction be made, and at what grade level?[4]

Footnote 2. Compare, for example, Z. P. Dienes, The Power of Mathematics, Hutchinson Educational Publishers, London, England (to appear).

Footnote 3. From a theoretical viewpoint, considerable prior informal study of algebra was included in grades 7 and 8, but the reality of actual practice usually worked out quite differently. In particular, where algebraic notions did appear in grade 8, they were most often presented via formal instruction that did not meet the criteria of "informal exploratory experience."

Footnote 4. One of the best discussions of the role of "informal exploratory experiences" is contained in: Jerome Bruner, The Process of Education, Harvard University Press, 1960.

The Objective of Madison Project Work -- Spring, 1961

In the spring of 1961 the Madison Project was engaged in working out, mainly at the Experimental Center in Weston, Connecticut, a program of "informal exploratory experiences" in algebra and geometry that were suitable for children in grades 2 through 8. These materials were presented to children for approximately one hour per week. The program at Weston had been in operation since the fall of 1959. Children who had begun the study of Madison Project materials in the fall of 1959 were -- in the spring of 1961 -- consequently completing their second year of study of Project materials.

The developmental work at Weston had centered on grades 4, 5, and 6. Materials developed in this experimentation had been assembled into a book, "Discovery in Algebra," written by Davis, and distributed by the Project at cost to interested school systems. In general, extensive use of "Discovery in Algebra" by teachers who had not studied Project materials (at, for example, a summer workshop) was not recommended, but some very small experimental use by teachers without special training was considered valuable, as it gave such teachers an opportunity to catch the flavor of Project materials without noticeably altering the child's experiences in school. This made it possible for teachers to make informed decisions as to whether they wished to attend some future Project workshops.

Achievement of Children -- Weston, Connecticut, Spring, 1961

Remembering that little or no formal instruction in algebra and geometry was given, it becomes a question of considerable importance to ask what sort of mathematical topics had been explored successfully by the children at Weston.

This can best be done by reviewing some specific topics studied by specific classes. Mathematics classes in grades 4, 5, and 6 at Weston are grouped homogeneously, decisions being made on the basis of the judgment of teachers and the guidance counsellor, who take account of class achievement, attitude, standardized test scores, and any other information available to them. Grouping is specifically for mathematics -- the Weston school has a semi-departmentalized organization in grades 4, 5, and 6. In spring, 1961, there were three sixth grade classes, of which we shall refer to the "top" class (the class strongest in mathematical achievement); there were three fifth-grade classes, of which we again refer to the top class; there were three fourth-grade classes, of which we shall refer to the top class.

All classes for these grades contained between 26 and 21 students, so we are discussing approximately the upper third of the entire student population at Weston for grades 4, 5, and 6. Weston is a small town (population 3,000), with a generally superior socio-economic situation. (There is, for example, a two-acre minimum on house lots. Most parents of both sexes are college graduates.) The average I.Q. (as measured by the school) for the sixth grade class was reported to be 126.6 using the Otis Quick Scoring Mental Ability Tests; standard deviation, 7.6.

Class A; finishing sixth grade, Spring, 1961. Class size 21; I.Q. (using the Otis Quick Scoring Mental Ability Tests) reported as 126.6; standard deviation, 7.6.

During the preceding year, grade 5, this class explored topics as follows:

i) discovered coefficient rules for quadratic equations,
ii) mastered arithmetic of signed numbers,
iii) worked out "equivalent equation" approach to linear equations,
iv) compiled a list of "identities" in algebra (such as $\square + \triangle = \triangle + \square$, $(\square + 3) \times (\square + 3) = (\square \times \square) + (6 \times \square) + 9$, etc.)
v) explored Cartesian co-ordinates and graphs of linear functions,
vi) solved word problems involving simultaneous inequalities, equations, and divisibility conditions,
vii) discovered graphical method for solving simultaneous linear equations,
viii) selected a set of axioms for the additive and multiplicative portions of arithmetic and algebra,
ix) proved many algebraic theorems from this set of axioms, including $\square + \square = 2 \times \square$, special cases of the binomial theorem, and theorems about the additive inverses of sums and products,
x) graphed circles and parabolas,
xi) explored linear and non-linear elasticity (following the work of Professor Robert Karplus, of the University of California at Berkeley).

During the academic year 1960-1961, grade 6, this class pursued a rather extensive study of matrix algebra and linear transformations. This year's work was successful, but somewhat less so than usual, presumably because somewhat too much formal pressure was exerted upon the children, and the atmosphere of "informal exploratory experience" was not adequately maintained. Achievements of the children this year included:

xii) discovery of "0" and "1" matrices for the system of 2 by 2 matrices,
xiii) exploration of matrices in relation to linear transformations in two dimensions,
xiv) use of linearity property to determine matrix inverses,
xv) exploration of isomorphic systems,
xvi) isomorphic representation of real numbers on a subset of the set of 2 by 2 matrices,
xvii) extension of number systems,
xviii) introduction of complex numbers by the use of 2 by 2 matrices,
xix) comparison of algebra of numbers with algebra of 2 by 2 matrices (commutativity, associativity, divisors of zero, etc.),
xx) discovery of vector representation of forces, velocities, and displacements,
xxi) discovery of zero-sum vector condition for equilibrium in statics,
xxii) exploration of similar triangles, scale drawings, and indirect measurement,
xxiii) indirect measurement of heights of flagpoles, ladders, etc.
xxiv) further exploration of word problems,
xxv) study of functions obtained empirically,
xxvi) development of an original unit on axiomatic geometry as an attempt to give an axiomatic description of the behavior of angles,
xxvii) study of "implication" and "contradiction,"
xxviii) derivation of mensuration formulae in geometry,
xxix) studied the "general form" of various equations.

Class B; finishing fifth grade, Spring, 1961. Class size 14; I.Q. (using the Otis Quick Scoring Mental Ability Tests) reported as 126.9; standard deviation, 5.6.

During the preceding year (1959-1960), Class B, then in grade 4, had pursued topics in algebra and geometry -- again, by informal independent exploration with virtually no exposition by the teacher -- as follows:

i) discovered coefficient rules for quadratic equations,
ii) mastered arithmetic of signed numbers,
iii) worked out "equivalent equation" approach to linear equations,
iv) explored the concepts of area, volume, and perimeter in 2 and 3 dimensions,

v) derived a few simple geometric mensuration formulae,
vi) solved certain simple equations in "general form" (i.e., with parametric coefficients),
vii) compiled a short list of algebraic identities,
viii) derived a few identities from other identities (in simple cases; e.g., derive A + (B x C) = (C x B) + A, given the commutative laws of addition and multiplication, the reflexive property of equality, and the replacement principle),
ix) explored Cartesian co-ordinates and graphs of linear functions,
x) explored a few kinds of systems of simultaneous linear equations, for example those with a coefficient matrix of the form

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

xi) explored linear and non-linear elasticity, following the work of Karplus.

During the academic year 1960-1961, Class B, in grade 5, pursued informal explorations into algebra and geometry as follows:

xii) explored graphs of circles and parabolas,
xiii) further exploration of identities in algebra,
xiv) derivation of additional mensuration formulae in geometry (2 and 3 dimensions),
xv) development of original axiomatic unit on the geometry of angles,
xvi) study of "implication" and "contradiction,"
xvii) further study of equations with parametric coefficients,
xviii) some little further study of derivations of algebraic identities,
xix) exploration of similar triangles, scale drawings, and indirect measurement,

Class C; finishing 4th grade, Spring, 1961. Class size 26. I.Q. (using Otis Quick Scoring Mental Ability Tests) reported as 123.3; standard deviation 5.8.

Class C began the study of Madison Project materials in the fall of 1960, when they entered fourth grade. (Some of the students had had a little previous exposure in third grade.) Topics covered included work with equations, graphs, and signed numbers similar to that described for Classes A and B.

Generally, it has been felt that the performance of Class B has shown less creativity and less originality than either Class A or Class C. Class A has made the most sophisticated, the most subtle, the most profound, and the most powerful explorations; Class C has been the most enthusiastic, and ranks close to Class A in originality and imagination.

The Description Problem - I

The work described above is most unusual. Much of the mathematics is "modern" and little known outside of college courses. The method of working with the children while they pursue their explorations is not a common one. The mathematical ability, understanding, and originality displayed by the children has few if any parallels elsewhere.

To make matters worse, the work is largely group work, so that the assessment of the contributions of individuals is a difficult (but not impossible) task. The objectives of this kind of school experience are largely intangible and hard to identify explicitly. Where they can be identified explicitly, they are hard to measure.

Further, the intrusion of "formal" educational elements must be resisted. Nearly all Project advisors feel that the high quality of student accomplishment would be seriously marred if the activity were not "fun," if the direction of the activity were too narrowly determined, and if the group of children were not allowed a sizeable measure of independence. Adults overtly passing judgment on these children would have a marked effect on the attitude of the children. In the informal atmosphere that is maintained, overt judgments of a formal nature are not passed on the work of the children, although the children do know that, in general, their work is highly regarded by the teacher.

In particular, the intrusion of written individual test papers would entirely change the nature of the teacher-student relationship in this kind of informal exploratory activity. The students do, of course, take ordinary written examinations (as individuals) on the school's regular arithmetic and science programs.

This implies that the problem of describing, measuring, or analyzing these informal explorations is a formidable one. Various approaches to the problem, with various ends in mind, will be described later in this report.

One important approach, however, has been to tape-record all class sessions and, as nearly as possible, all verbal exchanges between teachers and students before and after classes. Except for one or two occasions of equipment failure, tape recordings of all of these "group explorations" are available, and provide by far the most significant record of the Project's activities. Copies of these tape recordings are available to interested persons.

One tape recording, in particular, known as Tape No. D-1, has been widely circulated as a general introduction to the Project's activities; this tape was made by Class A during the academic year 1959-1960, when they were fifth graders. It is concerned with selecting a set of axioms for arithmetic and algebra.

In the spring of 1961 the Project consequently stood in approximately the condition described in the foregoing sections of this report.

The Task Ahead -- Spring, 1961

The tasks facing the Project in the spring of 1961 included these:

1) It was evidently necessary to follow Classes A, B, and C during ensuing years, since the real significance of these "informal exploratory experiences" in algebra and geometry could be assessed only in terms of the kind of secondary school mathematics program that might become possible for children with this background.

2) As mentioned above, the problem of describing, analyzing, and evaluating this kind of informal exploration was largely unsolved; one reason why some sort of description was necessary was the need to train additional teachers in the art of conducting such "informal explorations" with their own classes.

3) We might borrow from industry the phrase "quality control" to describe this next need. It includes attempts to answer these questions:

i) Is the Project activity harmful or beneficial to the (quite separate) arithmetic program? (That is, is student performance enhanced -- or otherwise -- on the school's usual arithmetic achievement tests?) It should be noted that improving student performance in traditional arithmetic was not an objective of the Madison Project program at this stage. (Nonetheless, the several schools using the Project materials at this stage did report significant improvement in arithmetic achievement.)

ii) Is the Project activity harmful or beneficial to the children's attitudes towards school, towards themselves, and towards their classmates?

Again -- for the most part -- improvement was not an objective of Project work at this stage, but the avoidance of negative effects was.

iii) Is the Project activity harmful or beneficial to the school program in areas other than arithmetic?

Once again, the immediate concern here is the avoidance of harmful consequences.

iv) Is the Project activity harmful or beneficial to general school morale (among faculty and administration)?

(Yet again the immediate concern is the avoidance of undesirable consequences.)

v) Is Teacher X doing a satisfactory job of providing "informal exploratory experiences" in mathematics to the children?

This question has two parts, one related to mathematical content, and the other related to the method of presentation.

vi) Since the mathematics studied is novel in varying degrees, the question arises: is this selection and presentation consistent with the child's continued growth in mathematics in the years ahead? (Both value judgments and empirical data-collection are involved here.)

4) The Project work in Weston had become highly valued in the eyes of children, parents, teachers, and administration. Consequently, a considerable need had arisen to find ways to expand this activity into more schools, in other communities.

5) The actual Project materials themselves needed to be extended and revised. In particular, as was noted earlier, if this kind of "informal exploratory experience" had as much value as it seemed to have, then it would be desirable to try to devise "informal exploratory experiences" in fractions for use with children in grades K-4 (or so), and in other topics for use at various points in the curriculum.

6) Indeed, if the apparent promise of this kind of educational activity should continue to bear fruit, the question necessarily arose: how large a part of the entire school program might someday be given over to "informal exploratory experiences" of this general sort?

7) Another kind of question also arose: what kinds of materials, in terms of physical format, could be provided to assist teachers and schools in providing "informal exploratory experiences" of this general sort? The usual sort of textbook might not be satisfactory, since it is predicated upon a general policy of "telling the answers" and "illustrating the procedures," which might be incompatible with "informal exploratory experiences."

One might turn to one or another form of programed learned materials, or to the use of file cards (as is done by William Hull at Shady Hill School, Cambridge, Massachusetts, by Z. P. Dienes, and others), or to a greater use of physical materials (as is done by W. W. Sawyer, Z. P. Dienes, Robert Karplus, Caleb Gattegno and others). Is there any possible use of motion picture films (to be shown in class to the children), or of educational television?

Up until this time the Project's classroom presentation had placed maximal reliance upon the teacher's knowledge and resourcefulness, and minimal reliance upon physical materials of any sort. Classroom activity was usually either "laboratory" experimentation with springs, measuring the height of a flagpole, etc., or -- more often -- consisted of "seminar" type discussion of more abstract matters (as can be heard on tape recording #D-1, alluded to earlier).

This, then, is approximately where the Project stood in the spring of 1961.

II. ACTIVITIES OF THE MADISON PROJECT FOR THE ACADEMIC YEAR 1961-1962

General Organization of the Project

For the academic year 1961-1962, the Project received financial support from the U. S. Office of Education, the National Science Foundation, the Ford Foundation, and a group of industries and trade unions in the greater St. Louis area. Total financial support was approximately $200,000. The Project undertook the following general activities:

1) Further development of Project materials (with NSF support);

2) Operation of Summer Institutes and In-Service Institutes, for the training of presently-active teachers in the use of Project materials (supported by NSF);

3) Collaboration with selected school systems to provide "informal exploratory experiences," using Project materials, for children; to attempt to define and analyze the nature of these "informal exploratory experiences"; to observe as fully as possible the consequences of such a program; and to communicate the results to the profession. (Supported by U. S. Office of Education Contract No. D-022);

4) The operation, in association with Webster College, of an entirely new program in undergraduate teacher education, intended to produce graduates who would be familiar with a thoroughly modern approach to the teaching and learning of mathematics, and who would be qualified to teach modern mathematics, and who would be qualified to teach modern mathematics at both elementary and secondary levels. (Supported by a Ford Foundation grant);

5) The production of a series of 16mm. sound motion pictures, showing actual classroom lessons. (Supported by a group of industries and trade unions in the vicinity of St. Louis, Missouri);

6) Co-operation with Educational Services, Incorporated (of Watertown, Massachusetts) in a program of contributions to African education in mathematics and science. (Supported by E.S.I., under grants from the Ford Foundation and the State Department);

It is, of course, with item 3 above that the present report is mainly concerned.

Participating schools: During the school year 1961-1962, the schools participating in this Demonstration were:

Weston, Connecticut, Public Schools,
Scarsdale, New York, Public Schools,
West Irondequoit, New York, Public Schools,
Clayton, Missouri, Public Schools,
Webster Groves, Missouri, Public Schools,
St. Louis (Missouri) County Catholic Parochial Schools,
Lexington, Massachusetts, Public Schools.

Project Personnel: During the school year 1961-1962, Project personnel included, in addition to those listed earlier, the following:

Co-ordinator for Webster College: Sister M. Jacqueline, S. L.,
Co-ordinator for Clayton, Missouri, Public Schools;
Harold Kaiser,
Demonstration teachers: Mr. Nicholas Sterling,
Sister Leann, S. L.,
Miss Lila Page.

In addition, Mr. J. Robert Cleary, Director of Program and
R search for the Scarsdale Public Schools, worked very closely with
t Project in relation to Scarsdale.

Activities Directed Toward Implementation: With regard to the "Demonstration" (item 3 on the list above), the Project undertook these activities:

i) It provided in-service training courses, 1 year in length (meeting 2 to 4 times monthly) for teachers in the participating schools.

ii) It provided a more intensive and extensive mutual training and exploring experience for the five persons who served as demonstration teachers (Davis, Sterling, Sister Leann, Mrs. Cochran, Miss Page).

iii) It conducted three meetings -- one on data collection, "quality control," description, analysis, and measurement (attended by mathematicians, psychologists, physicists, teachers, and educators); one on curriculum and school organization (attended by principals, superintendents, and supervisory personnel of participating schools); and one on the teaching and co-ordinating of Madison Project materials (attended by demonstration teachers and local co-ordinators).

iv) It made written materials, tape recordings, and films available to participating schools.

v) It operated a system of data collection and "quality control," as described later in this report.

vi) It provided consulting services for the participating schools, and operated a system of demonstration teaching and observation that will be described presently.

Activities of the Demonstration Teachers: Demonstration Teachers pl. one of two roles: they either work for the Project office, and serve as epresentatives to various schools, or else they serve as "local" co ordinators, and work mainly within a single school or a single school system. During 1961-1962, Davis, Sterling, and Sister Leann worked at the Project office and dealt with several schools; Mrs. Cochran served as local co-ordinator at Weston, Connecticut; and Miss Lila Page served as local co-ordinator at West Irondequoit, New York. Sterling served as consultant to the Webster Groves Public Schools, and to Mary Queen of Peace (a Catholic parochial school). Sister Leann served as consultant to two Catholic parochial schools. Davis served as consultant to Weston, Clayton, West Irondequoit, and to any interested schools in the area of Boston, Massachusetts (notably Lexington and Lincoln).

Mathematical Content

The mathematical content for Project materials is selected as follows:

First, a list of fundamental mathematical concepts and techniques is selected. (In practice, of course, this list is subject to continual modification.) This list includes:

variable,
open sentence and truth set,
classification of statements as "true" and "false"
locating points on Cartesian co-ordinates functions,
graph of a function, and graphical representation of
a truth set,
systems of simultaneous equations,
implication,
contradiction,
axioms, theorems, and derivations (mainly in algebra),
identities,
"general form" of equations; the use of variables in
writing "any" quadratic equation, "any" 2 x 2 matrix, etc.,
matrices,
vectors,
the arithmetic of signed numbers,
extensions of number systems,
isomorphism,
similar triangles
the trigonometric functions,
transformations,
linearity and non-linearity,
comparison of mathematical systems,
relation of abstract models to physical reality,
mean and standard deviation,
rate of change.

Specific mathematical "experiences" are now included in the Project materials if they satisfy four criteria:

1) They are directly related to the fundamental concepts and techniques on the list above.

2) The concept can be learned in a context of use by the child himself. Thus, "primes" would not be studied as a classification of integers in the sense of sterile classification; on the other hand, the concept of "prime" might appear as the child actively explored the general question of factoring integers, and would be useful in the factoring of all integers, whether prime or not.

In other words, the child plays an active role in solving problems; those ideas which he comes to use in his own efforts should form the basis of the mathematical concepts being learned.

3) There is opportunity for the child to "discover" some "patterns."[1]

4) The experience seems to be appropriate for children at the age in question.

There is another way to describe the selection of mathematical content, which involves successive approximations.

For the first approximation, assume that you wish to teach a very high-quality "good mathematics" algebra course in grade 9. What skills and understandings would you want your entering students to possess? This question forms the basis for the list of "fundamental concepts and techniques" that was given above.

As these materials are developed in use with children, it becomes possible to repeat this general approach, and thereby to arrive at subsequent approximations.

Examples of "Informal Exploratory" Lessons

Evidently, a knowledge of the "informal exploratory experiences" developed by the Project is fundamental to an understanding of the Project's activities.

By far the best way to learn about these "informal exploratory" lessons is by viewing various Project films, by visiting one of the experimental schools (especially Weston, Connecticut), or by listening to tape-recordings of lessons. Written descriptions are hardly satisfactory; nonetheless, four sample "informal exploratory lessons" might be described as follows:

i) A Lesson in Variables, Signed Numbers and the Classification of True Statements, False Statements, and Open Sentences.

Footnote 1. Cf. Robert B. Davis, The Madison Project -- A Brief Introduction to Materials and Activities, pp. 11-12.

After some preliminary work with open sentences such as

$$3 + \square = 5$$

and

$$3 + (2 \times \square) = 11,$$

the class is given a method for interpreting positive and negative integers in relation to double-entry bookkeeping, or to the number of pebbles-in-a-bag as compared with an arbitrary reference level.

At this point the class is asked to determine the truth set for the open sentence

$$(\square \times \square) - (5 \times \square) + 6 = 0;$$

the class is told that this truth set contains two elements, both positive integers.

By "trial-and-error," the class determines that the required truth set is

$$\{2, 3\}.$$

This example is followed by such examples as:

$$(\square \times \square) - (12 \times \square) + 35 = 0$$

$$(\square \times \square) - (8 \times \square) + 15 = 0$$

$$(\square \times \square) - (13 \times \square) + 22 = 0$$

$$(\square \times \square) - (19 \times \square) + 14 = 0$$

and so on.

This lesson illustrates quite well the four criteria listed earlier:

1) It is concerned with such basic concepts and techniques as variables, "substituting into equations," the arithmetic of signed numbers (addition and subtraction only), and the recognition of open sentences, true statements, and false statements.

2) The child is actively involved in problem-solving activity.

3) There is an opportunity for an interesting <u>discovery of pattern</u>: in the first quadratic equation listed above,

$$2 + 3 = 5, \quad 2 \times 3 = 6.$$

This same relation holds for all of the other quadratic equations listed, and, indeed, for <u>all</u> quadratic equations when written in this standard form.

4) The appropriateness of this lesson to children in grades 3 through 8 has been established in several hundred demonstration lessons, in tape recorded lessons, and in lessons on film.

(Three Project films deal with the lesson as described here:

"First Lesson"

"Sequencing and Elementary Ideas"

"Discovery."

In addition, the "pebbles-in-the-bag" model for signed numbers is presented in the film,

"A Lesson with Second Graders.")

ii) <u>A Lesson Dealing with Matrices, Isomorphism, Complex Numbers, and Extension of Number Systems</u>.

In this lesson, children who are somewhat familiar with matrix addition and multiplication, some properties of

matrices, and with simple linear and quadratic equations are asked to solve

$$\square \times \square = {}^{-}4.$$

This requires an extension of the number system, and the children are given one hint: to rewrite the equation (and other equations) in the language of 2-by-2 matrices.

That this lesson, also, satisfies the four criteria above can be easily verified.

(This lesson is presented in two Project films, one with 6th graders, and one with 7th graders.)

iii) A Lesson on Functions and Formulas. (This lesson follows a suggestion of Prof. W. W. Sawyer, of Wesleyan University, Middletown, Conn.)

Two or three children "make up a rule" -- such as "whatever number they tell you, double it and add seven." They do not reveal their rule to their classmates. However, their classmates call out numbers, and the two or three "experts" use their rule on each number suggested, and announce the answer. List the suggested numbers as replacements for "$\square$", and the "expert's" answers as replacements for "$\triangle$", to get a table:

$\square$	$\triangle$
1	9
2	11
0	7
10	27
11	29

etc.

Whoever first writes a formula for this function is entitled to become the "expert" and make up the next function.

An interesting and valuable situation occurs when two different formulas represent the same function, as with

$$(2 \times \square) + 6 = \triangle$$

$$(\square + 3) \times 2 = \triangle .$$

(This lesson can be used in grades 4, 5, 6, 7, and 8, with appropriate presentations.)

iv) Graphing Conic Sections.

Children who have had considerable experience with linear graphs (including the ideas of slope and intercept), and with graphs of circles and parabolas, are asked to graph

$$x^2 + 4y^2 = 25,$$

and thereafter to graph

$$x^2 + ky^2 = 25, \quad 0 \leqslant k.$$

This is their first experience with ellipses, and also their first experience with an equation involving a parameter.

This lesson, with a 7th grade class, is shown on the film:

"Graphing an Ellipse."

Achievements of the Children

Recalling that this problem-solving activity is a joint group activity by the entire class,: we recount class achievements during the academic year 1961-1962.

Class A (finishing 6th grade during the spring of 1961) Class size 21; I. Q. (Otis Quick Scoring Mental Ability Tests) reported as 126.6, with a standard deviation of 7.6. (The achievements of Class A prior to September, 1961, were discussed earlier in this report; see p. p. 6.)

Achievements during the academic year 1961-1962 (7th grade):

xxx) further study of the process of proving algebraic theorems from a set of axioms (extension of the set of axioms),

xxxi) classification of functions, and identification of functions from tabulated values,

xxxii) further study of linear and non-linear elasticity,

xxxiii) further study of matrices and complex numbers,

xxxiv) programing the IBM 1620 digital computer, using GOTRAN,

xxxv) derivation of the quadratic formula,

xxxvi) graphing $x^2 + ky^2 = 25, \quad 0 \leqq k$,

xxxvii) study of "implication" and "contradiction,"

xxxviii) use of vectors in physics; laboratory work combined with mathematical work.

Class B (finishing 5th grade during Spring, 1961). Class size 14; Otis I. Q. reported as 126.9 with a standard deviation of 5.6. (Class B was previously discussed in this report; cf. p. 7).

Achievements during the academic year 1961-1962 (6th grade):

xx) further study of derivations,

xxi) study of 2-by-2 matrices,

xxii) derivation of quadratic formula,

xxiii) further study of linear and non-linear elasticity,

xxiv) classification of functions, study of functions obtained empirically, etc.,

xxv) further study of "implication" and "contradiction,"

xxvi) preliminary study of $\square \times \square = 2$,

xxvii) programing the IBM 1620 digital computer, using GOTRAN,

xxviii) further experience deriving formulas for area, volume, and perimeter (2 and 3 dimensions).

Class C (described earlier in this report; cf. p. 8)

Achievements for the academic year 1961-1962 (5th grade): generally similar to that described for Class B. During 1961-1962, Class C continued to show a more creative, imaginative, and aggressive approach to mathematical thinking and mathematical exploration than did Class B. The reason for this is not known.

Analysis of Project Activity

The analysis of Project activity for the academic year 1961-1962 will be divided into several sections.

The first of these deals with the collection and preservation of data, mainly via tape recordings and film. The second, which we call the "description problem," deals with the need to draw defining boundaries around "informal exploratory experience," the role of the teacher in providing informal exploratory experiences, etc. The third, which we refer to as "quality control," deals with the need to identify possible desirable and undesirable consequences of Project activity. Finally, the fourth is concerned with the rationale behind the use of "informal exploratory experiences" as part of the child's experience in school.

The Teaching Problem - The Independent Variable

From the point of view of the classroom experience of the child, the independent variable is the experience provided by the school situation, and the dependent variable is the resulting development within the

child's behavior. Both of these variables are multi-dimensional: the school experience includes class size; composition (in terms of age, I.Q., achievement, attitude, temperament, etc.) of the student population in the class; temperament, mathematical knowledge, attitudes, and experience of the teacher; type of classroom activity; general atmosphere within the school; and so on. The resulting developments in the child's behavior include those relating to achievement in various areas, to attitude towards school, to enthusiasm for mathematics, to the child's interpretation of what constitutes mathematics and mathematical thinking, to the child's interpretation of his role as a student, and so on.

During the academic year 1961-1962, the Project's main concern was to achieve reasonable control or standardization of the independent variable. To put the matter in other language, the task for 1961-1962 was to achieve reasonable uniformity in providing "informal exploratory experiences." This matter will be discussed in some detail in the sections dealing with the "description problem" and the tape panel.

Data Collection

The ideal of data collection would be to record and preserve all data that might reveal any aspect of experimental activity -- not merely that which has been selected as relevant to a specific thesis. It should be possible for a subsequent investigator, bent on the exploration of entirely new theses, to draw his data from among that already collected. For example, a full description of the life of Beethoven should make it possible to determine whether, by twentieth century standards, Beethoven was in fact psychotic, even though many of the relevant concepts and criteria had not been developed during Beethoven's lifetime. (This task has, in fact, been attempted by several authors.)

A second desiderata in the collection of observational data -- and one which plagues physicist and behavioral scientist alike -- is the desirability that acts of observing and collecting data should not themselves influence the situation being observed. This is a particularly restrictive requirement in relation to informal educational experiences, the nature of which is altered entirely by the presence of any adult act of "passing judgment" on the behavior of the children.

Finally, data collection must be sufficiently specific that differing interpretations by different readers are avoided as much as possible.

The two main observational tools used by the Project during 1961-1962 were records of actual classroom lessons made on 16 mm. sound motion picture film, and similar records made on 1/4 inch audio tape, usually stereophonic.

The relevant technology need not be discussed here except to note that:

i) equipment is not concealed -- it is in plain view of the children,

ii) films are made by video tape recorder and kinescope technology, usually using four TV cameras,

iii) audio tapes are made with stereo Ampexes, using five microphones and appropriate mixers,

iv) some acoustical treatment of the room is made whenever possible.

The children readily adapt to the presence of the audio equipment, and their behavior is not noticeably different whether the equipment is present or not. The television cameras have slightly more effect, mainly because of the presence of camera men, the presence of high-intensity lighting, and because use of the TV equipment is less commonplace, but no important change can be observed in the children's behavior beyond that caused by the presence of any distracting element infringing upon the classroom scene (such as snow falling outdoors, or workmen in the school corridor, etc.). What is most important is that the informal nature of classroom activity is not altered by the presence of microphones and cameras.

Films. The film budget[1] of the Project for the interval in question was $50,000. With this budget it was possible to film approximately 20 hours of classroom lessons.

Footnote 1. This money was provided by a group of trade unions and industries in the vicinity of St. Louis, Missouri, through donations to Webster College.

This is a relatively small amount; consequently each film was intended to show a clearly defined exploration by the children. Two films show "first lessons," taught to children who had no previous experience with Madison Project materials (these are entitled "First Lesson," and "A Lesson with Second Graders"), one film shows the second hour of instruction on the day following "First Lesson" (this film is entitled "Second Lesson"), one film shows children confronted for the first time with a certain physical experiment involving vectors (this film is entitled "Forces and Vectors"), another shows children confronted for the first time with the task of graphing an ellipse (this film is entitled "Graphing an Ellipse"), and so on.

The children in these films are in grades 2 through 7, inclusive.

The task of making a film to show a specific lesson (as described above) creates problems in readiness: the children must have sufficient background experience to permit them to undertake the new lesson, but nothing must be done to cross the line that would render the "new" lesson a familiar one. Considerable thought is devoted to arranging this.

In general, many (but not all) of the achievements of the children, as catalogued earlier in this report, are documented on film.

The films have, among others, these values:

i) They record both teacher behavior and student behavior;

ii) They are admirably specific[1] -- one need not talk in general categories, such as "permissive" or "authoritarian" or "open-ended," but can refer specifically to "the kind of teaching shown in the 'Second Grade' film";

iii) They record data whose relevance may not be presently recognized -- nonetheless, this data is preserved for use in future studies;

Footnote 2. Cf. Kahn, R. L., and Cannell, C. F., The Dynamics of Interviewing (Wiley, 1957), p. 10, especially the quotation from D. Katz.

iv) They are valuable in teacher training;

v) They present data in a form that communicates directly to many diverse audiences: parents, teachers, school administrators, psychologists, psychiatrists, mathematicians, physicists, newspaper reporters, and so on;

vi) They are particularly appropriate to group problem-solving activity.

Tape Recordings. The audio tapes are generally similar to the films except that, because of their much smaller cost, audio tapes are used to record hundreds of hours of classroom lessons, in many different school systems. Visual effects are, of course, lost, but a great deal of data is nonetheless recorded. Most recordings are stereophonic.

The Description Problem - II

At least four things need better description:

i) What constitute "informal exploratory experiences"?

ii) What is the role of the teacher in this kind of teaching?

iii) What constitutes effective teaching in this context?

iv) What kind of behavior on the part of the student constitutes "good mathematical thinking"?

As one attack on this "description problem," the Project has developed a "tape panel," which we now describe:

The Tape Panel. The "tape panel" operates as follows: Fifty-two professional people, consisting of elementary school teachers, elementary team teacher leaders (for mathematics), secondary school mathematics teachers, elementary school principals, university mathematicians, psychologists, and psychiatrists serve as panelists. Each month several actual classroom lessons (or excerpts of lessons), in tape-recorded form, are circulated to the panelists. Each panelist writes

an individual analysis of the lessons, after which the 52 separate analyses are shared among all 52 panelists.

The analyses of the panelists show a surprising consensus on major points, and are contributing greatly to clearer definitions for the first 3 items listed above. Most of the operation of the tape panel will fall in the academic year 1962-1963, and will therefore be reported later. Nonetheless, certain preliminary remarks about the operation of the tape panel are included here.

1) Approximately one-third of the tape panelists write virtually identical analyses of most lessons;

2) It is possible to classify the panelists themselves, and variations among panelists within the same category are not so great as to pose problems;

3) In this way, a new attack on the problem of "objectivity" becomes possible;

4) If the one-third mentioned above are called the "reliable" interpreters of Project activities, objectives, and values, then this group of "reliable" observers can play a significant role in regard to "quality control"; this matter will be discussed later in this report.

Present plans call for the publication of a set of tape-recorded lesson excerpts, accompanied by a written volume of analysis. This can give great specificity to categories that have, in the past, often been too general to be meaningful.

What Constitutes an "Informal Exploratory Experience"?

Based partly upon analyses made by the tape panel, a preliminary description of an "informal exploratory experience" can be given as follows:

1) The class activity is lead by the teacher, but it is <u>not</u> dominated by the teacher;

2) Where possible, children are given success experiences;

3) The classroom atmosphere is such that "failure is non-punitive";

4) The teacher attempts to present each task to the class with reasonable clarity;

5) The teacher does not show the children how to attack the problem;

6) The teacher is reasonably responsive to student suggestions as to methods of attacking the problem, and even as to alternative objectives and alternative activities;

7) The teacher may base appeals upon objective reality, or upon the teacher's personal desires and preferences, but not upon authoritarian or formalistic criteria (e.g., the teacher may say "I'd rather do it the other way," or "Well, the other way looks like it might be easier"; he would not say "Now, how did I tell you to do this problem?");

8) The teacher formulates a flexible lesson plan in advance of the lesson, but he does not draw up a rigid lesson plan and attempt to adhere to it in spite of alternatives suggested by the students;

9) The teacher probably uses relatively informal language, and an informal manner;

10) The teacher listens to the children, and attempts to understand what they say. As Professor David Page and others have emphasized, it usually turns out that every student answer is at least partly correct, if one takes the trouble to identify the student's frame of reference.

11) The teacher treats the children with respect, not with condescension.

12) The teacher carefully avoids appearing omniscent; his statements nearly always suggest at least some measure of uncertainty in his own mind. Only in this way can he keep class activity directed toward the study of nature and of logical consequences, rather than toward authoritarian pronouncements.

It is possible for a program of "informal exploratory activities" to have a great cumulative effect; over a period of years there can be deep accumulated learning derived from such experiences.

"Informal exploratory experiences" may deal with relatively concrete matters (e.g., the use of Cuisenaire rods), or they may deal with quite abstract matters, such as

$\square \times \square = -4$ or $\square \times \square = 2$.

Factors in Teacher Effectiveness

Based partly upon analyses made by the tape panel, some preliminary discussion of teacher effectiveness can be given here.

Besides a knowledge of the relevant mathematics, 4 factors seem to exert a great influence on teacher effectiveness:

1) The effective teacher must believe that mathematics is discoverable. This is a subtle matter, and a mystifying difference in teacher effectiveness was largely cleared up by the remark of Professor Jerrold Zacharias of the M.I.T. Physics Department that "the scientifically oriented teacher regards science as a guessing game played against nature, against an underlying objective reality, whereas the misguided teacher builds guessing games against the mind of the teacher." The difference is quickly perceived by the children, and their response is quite different in the two cases. This may seem an obvious matter, but one of the hard tasks for the teacher is to distinguish that which "is discoverable" from that which "is not discoverable," and to treat each accordingly.

2) The effective teacher must believe in the child's ability to discover mathematics. Differences in teacher attitude on this point are clearly revealed in the tape-recorded lessons.

3) This third point requires more discussion. It has been possible to separate both teachers and panelists into two broad categories. Agreement within these categories is quite pronounced, but disagreement between the categories is always present. Dr. Brooks Fry and Dr. Carol Fry, of Los Angeles, have labelled these categories "ego functions dominated by

super-ego" and "ego functions dominated by id."

One might say that some teachers step into the classroom firmly mindful of "how a good child must behave," "how a good class must be conducted," "how a good teacher must behave," and so on. These are the teachers classified by the Frys as exhibiting "ego functions dominated by the super-ego."

Other teachers seem to bring to class a warm sympathy for their students, a perceptive intuitive feeling for human relations, a personal enthusiasm for mathematics, a conviction that the study of mathematics is fun, and so on. These, the teachers with "ego function dominated by id," are far more successful in teaching by "informal exploratory experiences" than are the teachers of the "super-ego" category.

One might say that the "id" group begin with reality and try to bring about growth, whereas the "super-ego" group begin with rather rigid abstract ideals, and try to compel conformity with these ideals.

4) A fourth factor seems to be the teacher's tolerance of "open-endedness." Scientific and mathematical explorations, if they are to be truly explorations, must necessarily have an "open-ended" quality, which causes discomfort to some teachers. Teachers who are uncomfortable when confronted with "open-endedness" are less able to make effective use of "in-tolerance" for "open-endedness."

The present remarks are, of course, preliminary. Further study of these matters will be the subject of tape panel analyses during the year 1962-1963.

The Ford Foundation Program at Webster College

In connection with the preceding remarks, it might be noted that the Madison Project collaborates with Webster College in the operation of a special undergraduate teacher education program, which has as one of its objectives the selection and education of young teachers with a strong "exploratory" approach to life and to learning and with a high tolerance of "open-endedness."

It has repeatedly been pointed out (as at the M.I.T.-Endicott House Conference, Summer, 1961) that significant progress in school mathematics

and science must await the creation of teacher education programs of this type.

Quality Control

In experimental activity on a large scale, a sharp look-out must be kept for possible desirable or undesirable consequences. This activity we have designated as "quality control," to emphasize that it is not primarily a matter of testing hypotheses.

The "quality control" for Project work during 1961-1962 has been divided into four parts:

1) Reliance upon the testing programs of the participating schools. Each participating school operates an extensive testing program of its own. None of these has revealed any cause for concern over undesirable consequences; in all cases, considerable growth in arithmetic skills is reported for students of the Project's "informal exploratory activities" in algebra and geometry. It should be repeated that attentiveness to possible undesirable consequences is important, but growth in arithmetic skills has not been a major goal of Project activity in 1961-1962.

2) Interviewing students and observing classes by school psychologists. Again, this is mainly a matter of being attentive to possible undesirable effects.

It might be remarked, informally, that the response of students and parents has, in fact, been dramatically enthusiastic. Reports of children who especially resist being absent on days of "math lessons" are very common. Mathematics is commonly reported to be these children's favorite subject.

3) Tape-recorded lesson excerpts by "Teacher X" can be submitted to the tape panel for evaluation. This activity will increase greatly during the year 1962-1963.

4) Tape-recorded and filmed lessons have been studied by a large number of prominent mathematicians, physicists, engineers, teachers, psychologists, psychiatrists, school administrators, and parents.

Some of the deepest questions are matters of value judgments: Are these children learning good mathematical thinking? Are these good choices of mathematical topics? What effects may result in the child's intellectual and emotional growth? Are these the kind of experiences that you want <u>your</u> children to get in school?

To attempt to answer questions of this nature by "impersonal" "objective" criteria would be foolhardy. To ignore such questions would invite serious errors. The Project has consequently relied upon the judgment of many of the nation's best-qualified experts. The list includes:[1]

Mathematicians:

Professor W. T. Martin, Chairman, Mathematics Department, Massachusetts Institute of Technology

Professor Arnold Ross, Chairman, Mathematics Department, Notre Dame University

Professor Donald E. Kibbey, Chairman, Mathematics Department, Syracuse University

Professor George Polya, Mathematics Department, Stanford University

Professor Saunders MacLane, Mathematics Department, University of Chicago

Professor E. E. Moise, Mathematics Department, Harvard University

Professor Andrew Gleason, Mathematics Department, Harvard University

Footnote 1. The Project does not claim the blanket endorsement of all individuals listed here, but it does have the benefit of their counsel and their opinions.

Professor Walter Prenowitz, Mathematics Department, Brooklyn College

Professor Wade Ellis, Oberlin College

Physicists:

Professor Francis Friedman, M.I.T.

Professor Jerrold Zacharias, M.I.T.

Professor Philip Morrison, Cornell University

Professor Melba Phillips, Washington University

Professor Robert Karplus, University of California (Berkeley)

Teachers:

Dr. Julius Hlavaty, DeWitt Clinton H.S., New York City

Psychologists:

Professor Jerome Bruner, Harvard University

Dr. Richard Anderson, East Brunswick, N.J., Public Schools

Professor Richard de Charms, Washington University

Psychiatrists:

Dr. Brooks Fry, M.D.

Dr. Carol Fry, M.D.

School Administrators:

Harold Howe, Supt. of Schools, Scarsdale, N. Y.

Dr. John Davis, Supt. of Schools, Lincoln, Massachusetts

Dr. David Robinson, Ass't. Supt. of Schools, Greece, N. Y.

Dr. Earl Helmer, Supt. of Schools, W. Irondequoit, N. Y.

J. Robert Cleary, Director of Curriculum and Research, Scarsdale Public Schools

Dr. Fred Roth, Scarsdale Public Schools

Educators:

Professor Earl Kelley, Wayne State University

Sister Marie Francis, S.L., Webster College

Sister M. Jacqueline, S.L., Webster College

School Psychologists:

Herbert Barrett, Weston, Conn., Public Schools

A particularly important forum for the discussion of "new" mathematics and science programs is provided by Educational Services, Incorporated, of Watertown, Massachusetts. E.S.I. assembles a large number of eminent scientists and educators for summer meetings at M.I.T.'s Endicott House, and for meetings during the academic year at M.I.T. and at Harvard. These meetings review current activity in the "new" mathematics and science programs. The Madison Project has been privileged to participate, and to present Project materials, at E.S.I.

meetings since 1961. The counsel available at E.S.I. meetings is one of the surest guarantees of the general appropriateness of Project materials. It should be remembered that much of the "quality control" in "new" mathematics programs is inescapably a matter of opinion, and the surest guarantee is, consequently, to secure the best opinion available.

Rationale for "Informal Exploratory Experiences"

The rationale supporting the inclusion of "informal exploratory experiences" within the school program has been presented in a sequence of important publications that have appeared within the last 40 years or so. Prominent among these are:

Hughes Mearns, "Creative Power"

Jerome Bruner, "Process of Education"

Brecher & Brecher, "Gifted Children Need Freedom to Learn"

Z. P. Dienes, "The Power of Mathematics"

A. S. Neill, "Summerhill"

Nathaniel Cantor, "The Dynamics of Learning"

Further strong evidence for the value of "informal exploratory experiences" is provided by certain laboratory experiments, notably those reported in:

> Harlow, H. F.; Harlow, M. K.; Meyer, D. R.; "Learning Motivated by a Manipulation Drive," Journal of Experimental Psychology, vol. 40 (1950), pp. 228-234

The Nature of Mathematical Thought

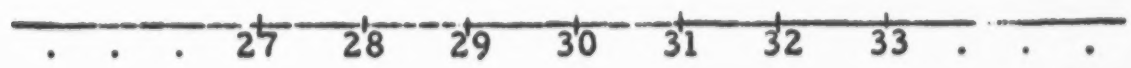
Evidently, any intelligently-conceived program to teach mathematics must start from some notion of what constitutes mathematical thought.

The Madison Project materials are based upon an interpretation of mathematical thought as consisting of at least the following ingredients:

1) Possibly most important - and most often neglected - the ability to discover patterns in abstract material. (This, for example, includes discovering the coefficient rules for quadratic equations, as mentioned earlier in this report.)

2) The ability to take an open-ended mathematical situation and to carry on intelligent exploration. (This, for example, includes taking the notion of "matrix multiplication" and extending this into a consideration of --say-- 2-by-2 matrices, the structure of the system of 2-by-2 matrices, isomorphism with real numbers, introduction of complex numbers and so on. [1])

3) Possession of a mental symbol system that will permit the student to perform "experiments" within his mind, by manipulating these symbols mentally, without recourse to external reality. (As one example, consider the question: Which is easier to remove first, shoes or stockings? One can answer this, without external experimentation, by manipulating certain mental symbols. As a second example, if one has a mental picture of the number line



one can answer the question: "How much is 29 plus 2?" by a similar "mental experiment" of "counting over two units" on one's internally-pictured number line. Similar mental symbols are used in playing chess without a board. [2])

Footnote 1. The Project films entitled "Matrices" and "Complex Numbers via Matrices" show children in grades 5, 6, and 7 pursuing this kind of exploration.

Footnote 2. Edward Chace Tolman, "Cognitive Maps in Rats and Men" - Chapter 19 of the volume, Behavior and Psychological Man, -- University of California Press, 1958.

4) An understanding of basic mathematical concepts, such as "variable," "function," "open sentence" and "truth set," the "number line," "isomorphism," "linearity," "Cartesian co-ordinates," "vector," "matrix," "transformation," "implication," etc.

5) A reasonable mastery of fundamental mathematical techniques, such as using variables, plotting points on Cartesian co-ordinates, performing arithmetical operations with the integers and with rational numbers, deriving algebraic theorems from a set of axioms, and so on.

6) A knowledge of certain facts, such as the fact that $^{-}1 \times {}^{-}1 = {}^{+}1$, the definition of the sine of an angle, etc.

Quite clearly, many "traditional" programs have placed emphasis mainly on items 5 and 6. These do not, in the Project view, make up the larger part of "mathematical thought."

A "New" Mathematics Curriculum

Evidently, the development of new mathematics curriculum is a matter of evolution. As with all evolution, the end is not clearly in sight -- evolution proceeds by continually modifying the existing situation, and the "end result" is not determined in advance.

Nonetheless, intelligent planning depends upon some reasonable assumptions about the future directions of this evolution.

At the present time, Madison Project materials are an incidental "enrichment" ingredient in a few school programs in grades 2-8 (or so).

The apparent success of this "enrichment" program suggests the possibility of a school curriculum on this general design:

Grades K-2	Mainly arithmetic
Grades 3, 4	Arithmetic, with beginnings of algebra and co-ordinate geometry

Grades 5, 6	Considerable study of abstract algebra and co-ordinate geometry
Grades 7, 8	Emphasis on physical experiments, descriptive geometry (scale drawing, etc.), logic
Grade 9-12	Algebra on the general pattern of Brumfiel, Charles, A Short Course in Algebra (available from the University of Michigan) Also, geometry (perhaps using both a vector approach and a co-ordinate approach), and other mathematical subjects on this level of sophistication.

Communications to the Profession

Evidently, an important aspect of a "demonstration" project is an effective flow of communication to the mathematical and educational community. The Madison Project work has been presented in some depth at meetings of the state mathematics teachers organizations in California, Connecticut, Oklahoma, New Jersey, New York, Wisconsin, Minnesota, Missouri, and Illinois. It has been presented at meetings of the National Council of Teachers of Mathematics, of the American Educational Research Association, of the Educational Records Bureau, of the Superintendents of Schools of New York State, of the National Association of Members of Boards of Education, of the National Education Association, and of several other organizations of elementary school principals, school administrators, etc. It has been presented at a large number of colleges and universities. Nearly all of these presentations have included showings of the Project films.

Publications

Project publications include:[1]

Discovery in Algebra

The Madison Project - A Brief Introduction to Materials and Activities

Matrices

A Lesson with Second Graders

First Lesson

Matrices, Functions, and Other Topics

Footnote 1: All publications on this list are available from the Madison Project, Webster College, St. Louis 19, Missouri

SUMMARY

A summary is included at the beginning of the present report, under the heading of "Abstract."

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